arXiv:0812.1135v1 [math.CA] 5 Dec 2008

KATZ'S MIDDLE CONVOLUTION AND YOKOYAMA'S EXTENDING OPERATION

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Abstract. We give a concrete relation between Katz's middle convolution and Yokoyama's extension and show the equivalence of both algorithms using these operations for the reduction of Fuchsian systems.

1. Introduction

Katz [Kz] introduces the operations *addition* and *middle convolution* of Fuchsian system

$$\frac{du}{dx} = \sum_{j=1}^{p} \frac{A_j}{x-t_j} u \tag{1.1}$$

of Schlesinger canonical form (SCF) on the Riemannian sphere and studies the rigid local systems. It has regular singularities at $x = t_1, \ldots, t_p$ and ∞. Here $A_j \in M(n,\mathbb{C})$ and $M(n,\mathbb{C})$ denotes the space of $n \times n$ matrices with entries in $\mathbb{C}$ and the number n is called the *rank* of the system. Katz shows that any irreducible rigid system of SCF is reduced to rank 1 system, namely a system with $n = 1$, by a finite iteration of these operations, which implies that any irreducible rigid system of SCF is obtained by applying a finite iteration of these operations to a rank 1 system since these operations are invertible.

The fact that the system is *rigid* is equal to say that it is free from accessory parameters but these operations are also useful for the study of non-rigid systems. In fact the Deligne-Simpson problem, the monodromies and integral representations of their solutions, their monodromy preserving deformations and their classification are studied by using these operations (cf. [DR2], [Ko], [HY], [HF], [O2] etc.).

Dettweiler and Reiter [DR] interpret these operations as those of tuples of matrices $\mathbf{A} = (A_1, \ldots, A_p)$ as follows.

The *addition* $M_\mu(\mathbf{A})$ of $\mathbf{A}$ with $\mu = (\mu_1, \ldots, \mu_p) \in \mathbb{C}^p$ is simply defined by

$$M_\mu(\mathbf{A}) = M_\mu^p(\mathbf{A}) := (A_1 + \mu_1, \ldots, A_p + \mu_p). \tag{1.2}$$

The *convolution* $(G_1, \ldots, G_p) \in M(pn,\mathbb{C})^p$ of $\mathbf{A}$ with respect to $\lambda \in \mathbb{C}$ is define by

$$\begin{aligned} G_j &:= \Big(\delta_{\mu,j}(A_\nu + \delta_{\mu,\nu}\lambda)\Big)_{\substack{1\le\mu\le p\\ 1\le\nu\le p}} \qquad (j = 1, \ldots, p) \\ &= \begin{array}{cc} & \begin{array}{ccccccc} & & & \overset{j}{\smile} & \phantom{A_{j+1}} & & \end{array} \\ j\,) & \begin{pmatrix} A_1 & A_2 & \cdots & A_j + \lambda & A_{j+1} & \cdots & A_p \\ & & & & & & \end{pmatrix} \end{array} \in M(pn,\mathbb{C}). \end{aligned} \tag{1.3}$$

Key words and phrases. *Fuchsian systems, middle convolution.*

2000 Mathematics Subject Classification. Primary 34M35; Secondary 34M40, 34M15

Supported by Grant-in-Aid for Scientific Researches (A), No. 20244008, Japan Society of Promotion of Science

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Since the subspaces

$$\mathcal{K} := \left\{ \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix} ;\ A_j u_j = 0 \quad (j = 1, \dots, p) \right\}, \tag{1.4}$$

$$\mathcal{L}_\lambda := \ker(G_1 + \cdots + G_p) \tag{1.5}$$

are G_j-invariant, we put $V := \mathbb{C}^{pn}/\mathcal{K}+\mathcal{L}_\lambda$ and define $\bar{G}_j \in \mathrm{End}(V) \simeq M(\dim V, \mathbb{C})$, which are the linear maps induced by G_j, respectively. Then the *middle convolution* $mc_\lambda(\mathbf{A})$ of $\mathbf{A}$ equals $(\bar{G}_1, \dots, \bar{G}_p)$.

For $\mathbf{A} = (A_1, \dots, A_p)$, $\mathbf{B} = (B_1, \dots, B_p) \in M(n, \mathbb{C})^p$ we write $\mathbf{A} \sim \mathbf{B}$ if there exists $g \in GL(n, \mathbb{C})$ such that $B_j = gA_jg^{-1}$ for $j = 1, \dots, p$ and we will sometimes identify $\mathbf{A}$ with $\mathbf{B}$ if $\mathbf{A} \sim \mathbf{B}$. The corresponding systems (1.1) will be also identified.

Yokoyama [Yo] introduces an extension and a restriction of the Fuchsian system

$$(xI_n - T)\frac{du}{dx} = Au \tag{1.6}$$

of Okubo normal form (ONF) with A, $T \in M(n, \mathbb{C})$ when T is a diagonal matrix and A satisfies a certain condition.

Suppose

$$T = \begin{pmatrix} t_1 I_{n_1} & & \\ & \ddots & \\ & & t_p I_{n_p} \end{pmatrix}, \tag{1.7}$$

where $n = n_1 + \cdots + n_p$ is a partition of n and $t_i \neq t_j$ if $i \neq j$. Put

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1p} \\ \vdots & \dots & \vdots \\ A_{p1} & \cdots & A_{pp} \end{pmatrix} \tag{1.8}$$

according to the partition, namely $A_{ij} \in M(n_i, n_j; \mathbb{C})$ which is the space of $n_i \times n_j$ matrices with entries in $\mathbb{C}$. Here we note that the system (1.6) of ONF is equal to the system (1.1) of SCF by putting

$$A_j := \begin{array}{c} \\ j) \\ \\ \end{array} \left(\begin{array}{cccc} & & & \\ A_{j1} & A_{j2} & \cdots & A_{jp} \\ & & & \end{array} \right) \in M(n, \mathbb{C}). \tag{1.9}$$

Conversely we have the following lemma.

Lemma 1.1. *Suppose* $(A_1, \dots, A_p) \in M(n, \mathbb{C})^p$ *satisfies*

$$\begin{cases} \mathrm{rank}\, A_1 + \cdots + \mathrm{rank}\, A_p = n, \\ \mathrm{Im}\, A_1 + \cdots + \mathrm{Im}\, A_p = \mathbb{C}^n. \end{cases} \tag{1.10}$$

Then there exists $g \in GL(n, \mathbb{C})$ *such that the* ν*-th row of* $g^{-1}A_jg$ *is identically zero if* $\nu \leq \mathrm{rank}\, A_1 + \cdots + \mathrm{rank}\, A_{j-1}$ *or* $\nu > \mathrm{rank}\, A_1 + \cdots + \mathrm{rank}\, A_j$. *Hence the system of SCF is equivalent to a system of ONF if* (1.10) *holds.*

Proof. The assumption implies that there exists a basis $\{v_1, \dots, v_n\}$ of $\mathbb{C}^n$ such that

$$\mathrm{Im}\, A_j = \sum_{\mathrm{rank}\, A_1 + \cdots + \mathrm{rank}\, A_{j-1} < \nu \leq \mathrm{rank}\, A_1 + \cdots + \mathrm{rank}\, A_j} \mathbb{C}v_j.$$

Then the expression of A_j under this basis has the required property, namely, we may put $g = (v_1, \dots, v_n) \in GL(n, \mathbb{C})$. $\square$

Remark 1.2. If a system (1.6) of ONF is linearly irreducible (cf. Definition 3.2), it satisfies (1.10) with (1.8) and (1.9).

Yokoyama [Yo] defines *extentions* $(\hat{T}, \hat{A}) = E_\epsilon(T, A)$ for $\epsilon = 0, 1$ and 2 with respect to two distinct complex numbers ρ_1 and ρ_2 when T and A_{ii} $(i = 1, \dots, p)$ are diagonalizable. Here ϵ is the number of the elements of $\{\rho_1, \rho_2\}$ which are not the eigenvalues of A.

Let

$$A_{ii} \sim \begin{pmatrix} \lambda_{i,1} I_{\ell_{i,1}} & & \\ & \ddots & \\ & & \lambda_{i,r_i} I_{\ell_{i,r_i}} \end{pmatrix} \tag{1.11}$$

with $\lambda_{i,j} \neq \lambda_{i,k}$ $(j \neq k)$ and $n_i = \ell_{i,1} + \cdots + \ell_{i,r_i}$ and fix a matrix $P \in GL(n, \mathbb{C})$ so that

$$A' := \begin{pmatrix} \mu_1 I_{m_1} & & \\ & \ddots & \\ & & \mu_q I_{m_q} \end{pmatrix} = P^{-1} A P \sim A, \tag{1.12}$$

where $n = m_1 + \cdots + m_q$ and $\mu_i \neq \mu_j$ $(i \neq j)$. Then $E_2(T, A) = (\hat{T}, \hat{A})$ with

$$\hat{T} := \begin{pmatrix} T & \\ & t_{p+1} I_n \end{pmatrix}, \tag{1.13}$$

$$\hat{A} := \begin{pmatrix} A & P \\ -(A' - \rho_1 I_n)(A' - \rho_2 I_n) P^{-1} & (\rho_1 + \rho_2) I_n - A' \end{pmatrix}. \tag{1.14}$$

When ρ_1 or ρ_2 is an eigenvalue of A, there exists a subspace invariant by $\hat{T}$ and $\hat{A}$ and the *extending operations* E_1 and E_0 of (T, A) are defined as follows. Putting

$$V_k := \Bigl\{ \begin{pmatrix} u \\ v_1 \\ v_2 \end{pmatrix} ;\ u \in \mathbb{C}^n,\ v_1 = 0 \in \mathbb{C}^k \text{ and } v_2 \in \mathbb{C}^{n-k} \Bigr\}. \tag{1.15}$$

we have

$$E_1(T, A) := (\hat{T}|_{V_{m_1}}, \hat{A}|_{V_{m_1}}) \qquad \text{when } \rho_1 = \mu_1, \tag{1.16}$$

$$E_0(T, A) := (\hat{T}|_{V_{m_1+m_2}}, \hat{A}|_{V_{m_1+m_2}}) \qquad \text{when } \rho_1 = \mu_1 \text{ and } \rho_2 = \mu_2. \tag{1.17}$$

Restrictions are defined as inverse operations of these extensions. It is proved by [Yo] that any irreducible rigid system of ONF with generic spectral parameters $\lambda_{i,j}$ and μ_k is reduced to a rank 1 system by a finite iteration of the extensions and restrictions and it gives the monodromy of the system.

In this note we clarify the direct relation between Yokoyama's operations and Katz's operations and then relax the assumption to define Yokoyama's operations (cf. Theorem 3.8 and Theorem 4.1). In particular we don't assume that the local monodromies of the system are semisimple (cf. Theorem 6.1). Moreover we show in Theorem 5.5 that the both operations on Fuchsian systems are equivalent in a natural sense. Hence the property of Katz's operation can be transferred to that of Yokoyama's operations and vice versa. For example, it is proved by [HF] that the middle convolution preserves the deformation equation and therefore so do Yokoyama's operations.

The author would like to express his sincere gratitude to Y. Haraoka and T. Yokoyama for valuable discussions.

2. Katz's middle convolution

For a partition $\mathbf{m} = (m_1, \ldots, m_N)$ of n with $n = m_1 + \cdots + m_N$ and $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ we define a matrix $L(\mathbf{m}; \lambda) \in M(n, \mathbb{C})$ as a representative of a conjugacy class, which is introduced and effectively used by [Os] (cf. [O2, §3]):

If $m_1 \geq m_2 \geq \cdots \geq 0$, then

$$L(\mathbf{m}; \lambda) := \Big(A_{ij}\Big)_{\substack{1 \leq i \leq N \\ 1 \leq j \leq N}}, \quad A_{ij} \in M(m_i, m_j, \mathbb{C}),$$

$$A_{ij} = \begin{cases} \lambda_i I_{m_i} & (i = j) \\ I_{m_i, m_j} := \Big(\delta_{\mu\nu}\Big)_{\substack{1 \leq \mu \leq m_i \\ 1 \leq \nu \leq m_j}} = \begin{pmatrix} I_{m_j} \\ 0 \end{pmatrix} & (i = j-1) \\ 0 & (i \neq j,\ j-1) \end{cases}. \tag{2.1}$$

For example

$$L(2, 1, 1; \lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} \lambda_1 & 0 & 1 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}.$$

Denoting $Z_{M(n,\mathbb{C})}(A) := \{X \in M(n, \mathbb{C})\, ;\, AX = XA\}$, we have

$$\dim \ker \prod_{j=1}^{k} \big(L(\mathbf{m}; \lambda) - \lambda_j\big) = m_1 + \cdots + m_k \quad (k = 1, \ldots, N), \tag{2.2}$$

$$\dim Z_{M(n,\mathbb{C})}\big(L(\mathbf{m}; \lambda)\big) = m_1^2 + \cdots + m_N^2. \tag{2.3}$$

In general we fix a permutation σ of indices $1, \ldots, N$ so that $m_{\sigma(1)} \geq m_{\sigma(2)} \geq \cdots$ and define $L(\mathbf{m}; \lambda) = L(m_{\sigma(1)}, \ldots, m_{\sigma(N)}; \lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(N)})$.

Let $\mathbf{A} = (A_1, \ldots, A_p) \in M(n, \mathbb{C})^p$. Put

$$A_0 = -(A_1 + \cdots + A_p). \tag{2.4}$$

Then Katz [Kz] defines

$$\operatorname{idx} \mathbf{A} := \sum_{j=0}^{p} \dim Z_{M(n,\mathbb{C})}(A_j) - (p-1)n^2, \tag{2.5}$$

which is called the *index of rigidity*.

If $\mathbf{A}$ is irreducible, $\operatorname{idx} \mathbf{A} \leq 2$. Moreover an irreducible $\mathbf{A}$ is rigid if and only if $\operatorname{idx} \mathbf{A} = 2$, which is proved by [Kz, §1.1.1]. Here $\mathbf{A}$ is called *irreducible* if any subspace V of $\mathbb{C}^n$ satisfying $A_j V \subset V$ for $j = 1, \ldots, p$ is $\{0\}$ or $\mathbb{C}^n$.

Using the representatives $L(\mathbf{m}; \lambda)$ of conjugacy classes of matrices, we can easily describes the property of the middle convolution.

Definition 2.1. For $\mathbf{A} \in M(n, \mathbb{C})^p$ we choose a tuple of $p + 1$ partitions $\mathbf{m} = (\mathbf{m}_0, \ldots, \mathbf{m}_p)$ and $\lambda_{j,\nu} \in \mathbb{C}$ so that

$$A_j \sim L\big(\mathbf{m}_j; \lambda_j\big) \text{ with } \mathbf{m}_j := (m_{j,1}, \ldots, m_{j,n_j}) \text{ and } \lambda_j := (\lambda_{j,1}, \ldots, \lambda_{j,n_j}) \tag{2.6}$$

for $j = 0, \ldots, p$. Here A_0 is determined by (2.4). We define the *Riemann scheme* of the corresponding system (1.1) of SCF by

$$\left\{ \begin{matrix} x = \infty & x = t_1 & \cdots & x = t_p \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{matrix} \right\}. \tag{2.7}$$

Here A_j is called the *residue matrix* of the system at $x = t_j$ $(j = 1, \dots, p)$ and A_0 is the residue matrix of the system at $x = \infty$. We also call (2.7) the Riemann scheme of $\mathbf{A}$. We will allow that some $m_{j,\nu}$ are 0.

Theorem 2.2 ([DR], [DR2])**.** *Let* $\mathbf{A} = (A_1, \dots, A_p) \in M(n, \mathbb{C})^p$. *Assume the following conditions:*

$$\bigcap_{\substack{1 \le \nu \le p \\ \nu \ne i}} \ker A_\nu \cap \ker(A_i + \tau) = \{0\} \qquad (i = 1, \dots, p,\ \forall \tau \in \mathbb{C}), \tag{2.8}$$

$$\sum_{\substack{1 \le \nu \le p \\ \nu \ne i}} \operatorname{Im} A_\nu + \operatorname{Im}(A_i + \tau) = \mathbb{C}^n \qquad (i = 1, \dots, p,\ \forall \tau \in \mathbb{C}). \tag{2.9}$$

Then $\bar{\mathbf{G}} = (\bar{G}_1, \dots, \bar{G}_p) := mc_\lambda(\mathbf{A})$ *satisfies* (2.8) *and* (2.9) *and*

$$\operatorname{idx} \bar{\mathbf{G}} = \operatorname{idx} \mathbf{A}. \tag{2.10}$$

If $\mathbf{A}$ *is irreducible, so is* $\bar{\mathbf{G}}$. *If* $\mathbf{A} \sim \mathbf{B}$, *then* $mc_\lambda(\mathbf{A}) \sim mc_\lambda(\mathbf{B})$. *Moreover we have*

$$mc_0(\mathbf{A}) \sim \mathbf{A}, \tag{2.11}$$

$$mc_{\lambda'} \circ mc_\lambda(\mathbf{A}) \sim mc_{\lambda'+\lambda}(\mathbf{A}). \tag{2.12}$$

Let (2.7) *be the Riemann scheme of* $\mathbf{A}$. *We may assume*

$$\begin{cases} \lambda_{0,1} = \lambda, & \\ \lambda_{i,0} = 0 & (i = 1, \dots, p), \\ \lambda_{j,\nu} = \lambda_{j,0} \ \Rightarrow\ m_{j,\nu} \le m_{j,0} & (\nu = 1, \dots, n_j,\ j = 0, \dots, p). \end{cases} \tag{2.13}$$

Note that $m_{j,0}$ *may be* 0. *Then the Riemann scheme*

$$\begin{Bmatrix} x = \infty & x = t_1 & \cdots & x = t_p \\ [\lambda]_{(m_{0,1})} & [0]_{(m_{1,1})} & \cdots & [0]_{(m_{p,1})} \\ [\lambda_{0,2}]_{(m_{0,2})} & [\lambda_{1,2}]_{(m_{1,2})} & \cdots & [\lambda_{p,2}]_{(m_{p,2})} \\ \vdots & \vdots & & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{Bmatrix} \tag{2.14}$$

of $\mathbf{A}$ *is transformed into the Riemann scheme*

$$\begin{Bmatrix} x = \infty & x = t_1 & \cdots & x = t_p \\ [-\lambda]_{(m_{0,1}-d)} & [0]_{(m_{1,1}-d)} & \cdots & [0]_{(m_{p,1}-d)} \\ [\lambda_{0,2} - \lambda]_{(m_{0,2})} & [\lambda_{1,2} + \lambda]_{(m_{1,2})} & \cdots & [\lambda_{p,2} + \lambda]_{(m_{p,2})} \\ \vdots & \vdots & & \vdots \\ [\lambda_{0,n_0} - \lambda]_{(m_{0,n_0})} & [\lambda_{1,n_1} + \lambda]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p} + \lambda]_{(m_{p,n_p})} \end{Bmatrix} \tag{2.15}$$

of $mc_\lambda(\mathbf{A})$ *with*

$$d = m_{0,1} + \cdots + m_{p,1} - (p-1)n. \tag{2.16}$$

Remark 2.3. If $\mathbf{A}$ is irreducible, then (2.8) and (2.9) are valid.

Suppose $\lambda \ne 0$. Since

$$\begin{aligned} &\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{pmatrix} \begin{pmatrix} A_1 + \lambda & \cdots & A_p \\ \vdots & \cdots & \vdots \\ A_1 & \cdots & A_p + \lambda \end{pmatrix} \\ &\qquad = \begin{pmatrix} A_1 + \lambda & \cdots & A_1 \\ \vdots & \cdots & \vdots \\ A_p & \cdots & A_p + \lambda \end{pmatrix} \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{pmatrix} \end{aligned} \tag{2.17}$$

and the linear map A_j induces the isomorphism $\mathbb{C}^n/\ker A_j \simeq \operatorname{Im} A_j$, we put

$$\tilde{A} := \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{pmatrix}, \tag{2.18}$$

$$G'_j := \begin{array}{rc} & \begin{array}{ccccccc} & & & \overset{j}{\smile} & & & \end{array} \\ j) & \begin{pmatrix} A_j & A_j & \cdots & A_j+\lambda & A_j & \cdots & A_j \end{pmatrix} \end{array} \in M(pn,\mathbb{C}) \tag{2.19}$$

for $j = 1, \dots, p$ and define

$$\begin{aligned} G'_0 &:= -(G'_1 + \cdots + G'_p), \\ \bar{G}'_j &:= G'_j\big|_{\operatorname{Im} \tilde{A}/\ker G'_0} \qquad (j = 0, \dots, p), \end{aligned} \tag{2.20}$$

Lemma 2.4. *Suppose $\lambda \neq 0$ and put $G_0 = -(G_1 + \cdots + G_p)$. Then under the above notation*

$$\tilde{A}G_j = G'_j\tilde{A} \qquad\qquad (j = 0, \dots, p), \tag{2.21}$$

$$G'_j \operatorname{Im} \tilde{A} \subset \operatorname{Im} \tilde{A}, \quad \ker G'_0 = \bigcap_{j=1}^{p} \ker G'_j \qquad\qquad (j = 0, \dots, p), \tag{2.22}$$

$$\tilde{A}(\mathcal{K} + \mathcal{L}_\lambda) = \ker G'_0 \subset \operatorname{Im} \tilde{A}, \tag{2.23}$$

and therefore $\tilde{A} \in \operatorname{End}(\mathbb{C}^{pn})$ induces the isomorphism

$$\begin{aligned} (\bar{G}_1, \dots, \bar{G}_p) = mc_\lambda(A_1, \dots, A_p) \quad &\in \Big(\operatorname{End}\big(\mathbb{C}^{pn}/\mathcal{K} + \mathcal{L}_\lambda\big)\Big)^p \\ \sim (\bar{G}'_1, \dots, \bar{G}'_p) \qquad &\in \Big(\operatorname{End}\big(\operatorname{Im} \tilde{A}/\ker G'_0\big)\Big)^p. \end{aligned} \tag{2.24}$$

In particular if $-\lambda$ is not the eigenvalue of $A_1 + \cdots + A_p$, the middle convolution $mc_\lambda(\mathbf{A})$ transforms the system (1.1) *of SCF to the system of ONF*

$$\left(xI_{n'_1+\cdots+n'_p} - \begin{pmatrix} t_1 I_{n'_1} & & \\ & \ddots & \\ & & t_p I_{n'_p} \end{pmatrix} \right) \frac{du}{dx} = \big(-G'_0\big|_{\operatorname{Im} A_1 \oplus \cdots \oplus \operatorname{Im} A_p}\big)u \tag{2.25}$$

with $n'_j = \dim \operatorname{Im} A_j$.

Proof. Note that $\tilde{A}G_0 = G'_0\tilde{A}$, which corresponds to (2.17), and moreover that (2.21) and (2.22) are also clear.

Since $\mathcal{K} = \ker \tilde{A}$ and $\mathcal{L}_\lambda = \ker G_0$, $G'_0\tilde{A}(\mathcal{K}+\mathcal{L}_\lambda) = G'_0\tilde{A}\ker G_0 = \tilde{A}G_0 \ker G_0 = 0$ and therefore $\tilde{A}(\mathcal{K} + \mathcal{L}_\lambda) \subset \ker G'_0$. Let $u \in \ker G'_0$. Putting

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix}, \quad u_j \in \mathbb{C}^n, \quad v := u_1 + \cdots + u_p \ \text{ and } \ \tilde{v} := \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix},$$

we have $\lambda u_j = -A_j v$ and therefore $\lambda v + (A_1 + \cdots + A_p)v = 0$. Hence $\tilde{v} \in \ker G_0$ and $u = -\lambda^{-1}\tilde{A}\tilde{v} \in \tilde{A}\ker G_0$. Thus we have (2.23). $\square$

3. Yokoyama's extending operation

First we examine the conditions (2.8) and (2.9) for the Fuchsian system (1.6) of ONF with (1.7).

For a partition $n = k_1 + \cdots + k_q$ and $C_j \in M(k_j, \mathbb{C})$ we denote

$$\operatorname{diag}(C_1, \ldots, C_q) := \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_p \end{pmatrix} \in M(n, \mathbb{C}),$$
$$O_{k_j} := 0 \in M(k_j, \mathbb{C}).$$

Then A_j given by (1.9) equals $\operatorname{diag}(O_{n_1+\cdots+n_{j-1}}, I_{n_j}, O_{n_{j+1}+\cdots+n_p})A$.

Lemma 3.1. *The pair of conditions* (2.8) *and* (2.9) *for* A_j *given by* (1.9) *is equivalent to the pair of conditions*

$$\operatorname{rank} A = n \tag{3.1}$$

and

$$\begin{cases} \operatorname{rank}\big((A+\tau)\operatorname{diag}(O_{n_1+\cdots+n_{i-1}}, I_{n_i}, O_{n_{i+1}+\cdots+n_p})\big) = n_i, \\ \operatorname{rank}\big(\operatorname{diag}(O_{n_1+\cdots+n_{i-1}}, I_{n_i}, O_{n_{i+1}+\cdots+n_p})(A+\tau)\big) = n_i \end{cases} \tag{3.2}$$
$$\textit{for any } \tau \in \mathbb{C} \textit{ and } j = i, \ldots, p.$$

Proof. Note that the condition (2.8) with $\tau = 0$ equals (3.1), which implies (3.2) with $\tau = 0$.

Suppose $\tau \neq 0$ and (3.1). Put $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix}$ with $u_\nu \in \mathbb{C}^{n_\nu}$. Then $\sum_{\nu \neq i} \operatorname{Im} A_\nu = \{\mathbf{u} \in \mathbb{C}^n\,;\, u_i = 0\}$ and therefore the condition (2.9) is equivalent to the second condition of (3.2). Since $\ker(A_i + \tau) = \{\mathbf{u} \in \mathbb{C}^n\,;\, (A_{ii}+\tau)u_i = 0 \text{ and } u_\nu = 0 \quad (\nu \neq i)\}$, the condition (2.8) is equivalent to the condition $\{u_i \in \mathbb{C}^{n_i}\,;\, (A_{ii}+\tau)u_i = 0 \text{ and } A_{\nu,i}u_i = 0 \quad (\nu \neq i)\} = \{0\}$, which is equivalent to the second condition of (3.2). □

Definition 3.2. The system (1.1) of SCF is called *linearly irreducible* if A_j have no non-trivial common invariant subspace of $\mathbb{C}^n$, namely, $\mathbf{A} = (A_1, \ldots, A_p)$ is irreducible. Then

$$\text{irreducible} \;\Rightarrow\; \text{linearly irreducible} \;\Rightarrow\; (3.1) \text{ and } (3.2). \tag{3.3}$$

Remark 3.3. The conditions (3.1) and (3.2) are valid if the system (1.6) of ONF is irreducible as a differential equation or linearly irreducible.

Assume (3.1) and (3.2) for the system (1.6) of ONF. Put $\lambda = -\rho_1 \neq 0$ and apply Lemma 2.4 to $\mathbf{A} = (A_1, \ldots, A_p)$ given by (1.9). Then (3.1) assures $\operatorname{Im} A_j \simeq \mathbb{C}^{n_j}$. Under the notation in the proof Lemma 3.1 the projection defined by

$$\iota_j : \mathbb{C}^n \ni \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix} \mapsto u_j \in \mathbb{C}^{n_j}$$

gives this isomorphism and hence we have the isomorphism $\iota : \operatorname{Im} \tilde{A} \simeq \mathbb{C}^{n_1+\cdots+n_p} = \mathbb{C}^n$. Under the identification of this isomorphism ι we have

$$
G'_j|_{\operatorname{Im}\tilde{A}} \simeq G''_j := \begin{array}{c c} & \begin{array}{ccccccc} \phantom{A_{j1}} & \phantom{A_{j2}} & & \overset{j}{\smile} & \phantom{A_{jj+1}} & & \phantom{A_{jp}} \end{array} \\ j\,) & \begin{pmatrix} A_{j1} & A_{j2} & \cdots & A_{jj}-\rho_1 & A_{jj+1} & \cdots & A_{jp} \end{pmatrix} \end{array} \in M(n,\mathbb{C}),
$$

$$
\begin{aligned}
&G''_1 + \cdots + G''_p = A - \rho_1, \\
&G'_j|_{\operatorname{Im}\tilde{A}/\ker G'_0} \simeq \bar{G}''_j := G''_j|_{\mathbb{C}^n/\ker(A-\rho_1)}
\end{aligned}
$$

for $j = 1, \dots, p$ and

$$
\begin{aligned}
(3.4)\quad mc_{-\rho_1}(A_1,\dots,A_p) &\in \Big(\operatorname{End}\big(\mathbb{C}^{pn}/\mathcal{K}+\mathcal{L}_\lambda\big)\Big)^p \\
&\sim (\bar{G}''_1,\dots,\bar{G}''_p) \in \Big(\operatorname{End}\big(\mathbb{C}^n/\ker G'_0\big)\Big)^p.
\end{aligned}
$$

In particular we have

Corollary 3.4. *Suppose the system* (1.6) *of ONF satisfies* (3.1) *and* (3.2). *If* $-\lambda$ *is not the eigenvalue of* A, *then the middle convolution* $mc_\lambda(A_1,\dots,A_p)$ *corresponds to the transformation* $A \mapsto A+\lambda$ *of the system* (1.6).

Definition 3.5. We denote this operation of the system of ONF by E_λ and call it a *generic Euler transformation*, which is defined if $-\lambda$ is not the eigenvalue of A. Note that $E_\lambda \circ E_{\lambda'} = E_{\lambda+\lambda'}$.

The transformation $A \mapsto A+\lambda$ of (1.6) corresponds the Riemann-Liouville integral

$$
I_t^\lambda u(x) := \frac{1}{\Gamma(\lambda)} \int_t^x (x-s)^{\lambda-1} u(s)\,ds \tag{3.5}
$$

of the solution $u(x)$ of the system (cf. [Kh, Chapter 5]). Here $t \in \{t_1,\dots,t_p,\infty\}$.

Definition 3.6. Define the linear maps

$$
\begin{array}{cccc}
T_{(j,\infty)}: & M(n,\mathbb{C})^p & \rightarrow & M(n,\mathbb{C})^p \\
& \cup & & \cup \\
& \big(B_1,\dots,B_p\big) & \mapsto & \big(B_1,\dots,B_{j-1},-(B_1+\cdots+B_p),B_{j+1},\dots,B_p\big)
\end{array}
$$

for $j = 1, \dots, p$ and

$$
\begin{array}{cccc}
T_\sigma: & M(n,\mathbb{C})^p & \rightarrow & M(n,\mathbb{C})^p \\
& \cup & & \cup \\
& \big(B_1,\dots,B_p\big) & \mapsto & \big(B_{\sigma(1)},\dots,B_{\sigma(p)}\big).
\end{array}
$$

Here σ is a permutation of the indices $1,\dots,p$. Under the natural identification

$$
M(n,\mathbb{C})^p \simeq \{(B_1,\dots,B_{p+1}) \in M(n,\mathbb{C})^{p+1}\,;\, B_{p+1} = 0\} \subset M(n,\mathbb{C})^{p+1} \tag{3.6}
$$

we have $T_{(p+1,\infty)}(B_1,\dots,B_p) = (B_1,\dots,B_p,-(B_1+\cdots+B_p))$.

Remark 3.7. i) Let $\mathbf{B} \in M(n,\mathbb{C})^p$. Then $T_{(p+1,\infty)}\mathbf{B}$ is irreducible if and only if $\mathbf{B}$ is irreducible. Similarly $T_{(p+1,\infty)}\mathbf{B}$ satisfies (2.8) and (2.9) if and only if so does $\mathbf{B}$.

ii) The map $T_{(p+1,\infty)}$ corresponds to the transformation of the Fuchsian system of SCF induced from the automorphism of the Riemannian sphere defined by $x \mapsto \frac{t_{p+1}x-c}{x-t_{p+1}}$. Here $c \in \mathbb{C}$, $c \neq t_{p+1}^2$ and $t_{p+1} \neq t_j$ for $j = 1,\dots,p$.

iii) The middle convolution mc_λ clearly commutes with T_σ, namely,

$$
mc_\lambda \circ T_\sigma = T_\sigma \circ mc_\lambda. \tag{3.7}
$$

Fix $\rho_2 \neq 0$ and examine $mc_{\rho_1} \circ M_{(0,\dots,0,\rho_2-\rho_1)} \circ T_{(p+1,\infty)} \circ mc_{-\rho_1}(A_1,\dots,A_p)$.

Since $G''_j \ker(A-\rho_1) = 0$, it follows from (3.2) that $\operatorname{Im} \bar{G}''_j = \operatorname{Im} G''_j / G''_j \ker(A-\rho_1) \simeq \mathbb{C}^{n_j}$. Note that

$$\begin{aligned}
M_{(0,\dots,0,\rho_2-\rho_1)} \circ T_{(p+1,\infty)}(\bar{G}''_1,\dots,\bar{G}''_p) &= (\bar{G}''_1,\dots,\bar{G}''_p, -\bar{G}''_1 - \cdots - \bar{G}''_p + \rho_2 - \rho_1) \\
&= \big(\bar{G}''_1,\dots,\bar{G}''_p, (-A+\rho_2)\big|_{\mathbb{C}^n/\ker(A-\rho_1)}\big), \\
\bar{V} := \operatorname{Im}(\bar{G}''_1 + \cdots + \bar{G}''_p + \rho_1 - \rho_2) &= \operatorname{Im}(A-\rho_2)/(A-\rho_2)\ker(A-\rho_1) \\
&= \begin{cases} \operatorname{Im}(A-\rho_2)/\ker(A-\rho_1) & (\rho_1 \neq \rho_2) \\ \operatorname{Im}(A-\rho_2) & (\rho_1 = \rho_2) \end{cases}
\end{aligned}$$

and $(\bar{G}''_1,\dots,\bar{G}''_p, \bar{G}''_1 + \cdots + \bar{G}''_p + \rho_1 - \rho_2)$ satisfies the conditions corresponding to (2.8) and (2.9).

Moreover we remark that the last claim on the conditions corresponding to (2.8) and (2.9) doesn't necessarily imply that $(\bar{G}''_1,\dots,\bar{G}''_p)$ satisfies the conditions.

Applying Lemma 2.4 to $mc_{\rho_1}(\bar{G}''_1,\dots,\bar{G}''_p, -\bar{G}''_1 - \cdots - \bar{G}''_p + \rho_2 - \rho_1)$, we have

$$mc_{\rho_1}(\bar{G}''_1,\dots,\bar{G}''_p, -\bar{G}''_1 - \cdots - \bar{G}''_p + \rho_2 - \rho_1) \sim (A'_1,\dots,A'_p, A'_{p+1}),$$

$$A'_j = \begin{array}{c} \\ j) \end{array} \!\! \begin{array}{c} \begin{array}{ccccccccc} & & & & & \overbrace{}^{p+j} & & \end{array} \\ \begin{pmatrix} A_{j1} & \cdots & A_{jp} & A_{j1} & \cdots & A_{jj}-\rho_1 & \cdots & A_{jp} \end{pmatrix} \end{array} \in M(2n,\mathbb{C}),$$

$$A'_{p+1} = \begin{pmatrix} O_n & O_n \\ -A+\rho_2 & -A+\rho_1+\rho_2 \end{pmatrix} \in M(2n,\mathbb{C})$$

for $j = 1,\dots,p$. Here A'_j and A'_{p+1} are endomorphisms of the linear space

$$U := \Big\{ \begin{pmatrix} u \\ v \end{pmatrix} \,;\, u \in \mathbb{C}^n,\ v \in \bar{V} \Big\}. \tag{3.8}$$

Since

$$A'_1 + \cdots + A'_{p+1} = \begin{pmatrix} A & A-\rho_1 \\ -A+\rho_2 & -A+\rho_2+\rho_1 \end{pmatrix} \tag{3.9}$$

and

$$\begin{aligned}
&\begin{pmatrix} I_n & \\ & A-\rho_1 \end{pmatrix} \begin{pmatrix} A & A-\rho_1 \\ -A+\rho_2 & -A+\rho_1+\rho_2 \end{pmatrix} \\
&\qquad = \begin{pmatrix} A & I_n \\ -(A-\rho_1)(A-\rho_2) & -A+\rho_1+\rho_2 \end{pmatrix} \begin{pmatrix} I_n & \\ & A-\rho_1 \end{pmatrix}
\end{aligned} \tag{3.10}$$

and $A-\rho_1 : \bar{V} \xrightarrow{\sim} \operatorname{Im}(A-\rho_1)(A-\rho_2)$, we have

$$mc_{\rho_1}(\bar{G}''_1,\dots,\bar{G}''_p, -\bar{G}''_1 - \cdots - \bar{G}''_p + \rho_2 - \rho_1) \sim (\hat{A}_1,\dots,\hat{A}_p,\hat{A}_{p+1})$$

with

$$\hat{A} := \begin{pmatrix} A & I_n \\ -(A-\rho_1)(A-\rho_2) & -A+\rho_1+\rho_2 \end{pmatrix} \in \operatorname{End}(\bar{V}), \tag{3.11}$$

$$\begin{aligned}
\hat{V} &:= \mathbb{C}^n \oplus \operatorname{Im}(A-\rho_1)(A-\rho_2) \\
&= \Big\{ \begin{pmatrix} u \\ v \end{pmatrix} \,;\, u \in \mathbb{C}^n,\ v \in \operatorname{Im}(A-\rho_1)(A-\rho_2) \Big\} \subset \mathbb{C}^{2n},
\end{aligned} \tag{3.12}$$

$$\hat{A}_j := \operatorname{diag}(O_{n_1+\cdots+n_{j-1}}, I_{n_j}, O_{n_{j+1}+\cdots+n_p+n})\hat{A} \qquad (j = 1,\dots,p), \tag{3.13}$$

$$\hat{A}_{p+1} := \operatorname{diag}(O_n, I_n)\hat{A}. \tag{3.14}$$

Thus we have the following theorem.

Theorem 3.8 (Extending operation). *Suppose that the Fuchsian system* (1.6) *of ONF satisfies* (3.1) *and* (3.2). *Then for any complex numbers* ρ_1 *and* ρ_2 *with* $\rho_1\rho_2 \neq 0$, $(\hat{A}_1, \dots, \hat{A}_{p+1}) := mc_{\rho_1} \circ M^{p+1}_{(0,\dots,0,\rho_2-\rho_1)} \circ T_{(p+1,\infty)} \circ mc_{-\rho_1}(A_1, \dots, A_p)$ *defines a Fuchsian system*

$$(xI_{\hat{n}} - \hat{T})\frac{du}{dx} = \hat{A}u \tag{3.15}$$

of ONF satisfying (3.1) *and* (3.2). *Here* $\hat{T} = \operatorname{diag}(t_1 I_{n_1}, \dots, t_p I_{n_p}, t_{p+1} I_{n_{p+1}}) \in \operatorname{End}(\hat{V})$, $\hat{V} \simeq \mathbb{C}^{\hat{n}}$ *and* $\hat{A} \in \operatorname{End}(\hat{V})$ *are defined by* (3.11) *and* (3.12) *and*

$$\hat{n} = \dim \hat{V} = n + n_{p+1}, \quad n_{p+1} = \dim \operatorname{Im}(A - \rho_1)(A - \rho_2). \tag{3.16}$$

Moreover (3.15) *is linearly irreducible if and only if* (1.6) *is linearly irreducible.*

Let

$$\begin{Bmatrix} x = \infty & x = t_1 & \cdots & x = t_p \\ [-\mu_1]_{(m_1)} & [0]_{(n-n_1)} & \cdots & [0]_{(n-n_p)} \\ [-\mu_2]_{(m_2)} & [\lambda_{1,1}]_{(\ell_{1,1})} & \cdots & [\lambda_{p,1}]_{(\ell_{p,1})} \\ \vdots & \vdots & & \vdots \\ [-\mu_q]_{(m_q)} & [\lambda_{1,r_1}]_{(\ell_{1,r_1})} & \cdots & [\lambda_{p,r_q}]_{(\ell_{p,r_q})} \end{Bmatrix} \tag{3.17}$$

be the Riemann scheme of the system (1.6) *of ONF, which is compatible with the notation in* (1.11) *and* (1.12) *etc. when* A_{ii} *and* A *are diagonalizable. We may assume*

$$\begin{cases} \rho_1 = \mu_1 \ \ \text{and} \ \ \rho_2 = \mu_2, \\ \mu_\nu = \rho_1 \ \Rightarrow \ m_\nu \le m_1, \\ \mu_\nu = \rho_2 \ \text{and} \ \nu > 1 \ \Rightarrow \ m_\nu \le m_2. \end{cases} \tag{3.18}$$

Here m_1 *and* m_2 *may be* 0. *Then the Riemann scheme of the system* (3.15) *equals*

$$\begin{Bmatrix} x = \infty & x = t_1 & \cdots & x = t_p & x = t_{p+1} \\ [-\mu_1]_{(n-m_2)} & [0]_{(\hat{n}-n_1)} & \cdots & [0]_{(\hat{n}-n_p)} & [0]_{(n)} \\ [-\mu_2]_{(n-m_1)} & [\lambda_{1,1}]_{(\ell_{1,1})} & \cdots & [\lambda_{p,1}]_{(\ell_{p,1})} & [\mu_1 + \mu_2 - \mu_3]_{(m_3)} \\ & \vdots & & \vdots & \vdots \\ & [\lambda_{1,r_1}]_{(\ell_{1,r_1})} & \cdots & [\lambda_{p,r_q}]_{(\ell_{p,r_q})} & [\mu_1 + \mu_2 - \mu_q]_{(m_q)} \end{Bmatrix} \tag{3.19}$$

with $\hat{n} = 2n - m_1 - m_2$.

Remark 3.9. i) Suppose that the system (1.6) satisfies (3.1) and (3.2). Then

$$q \ge 2, \tag{3.20}$$

$$\mu_\nu \neq 0 \qquad (\nu = 1, \dots, q), \tag{3.21}$$

$$\ell_{j,\nu} \le n - n_j \qquad (\nu = 1, \dots, r_j, \ j = 1, \dots, p), \tag{3.22}$$

$$m_\nu \le \min\{n_1, \dots, n_p\} \qquad (\nu = 1, \dots, q) \tag{3.23}$$

under the notation in the Theorem 3.8. For example the condition $\ker(A_j - \lambda_{j,\nu}) \cap \bigcap_{\nu \neq j} \ker A_\nu = \{0\}$ with $\dim \ker A_\nu = n_\nu$ assures (3.22).

ii) Yokoyama [Yo] defines the extending operation for generic parameters $\lambda_{j,\nu}$, μ_ν, ρ_1 and ρ_2. It is assumed there that A_{ii}, A, $\hat{A}_{ii}$ and $\hat{A}$ are diagonalizable, $\operatorname{rank} A_{ii} = n_i$, $\rho_1 \neq \rho_2$ etc. In this note we don't assume these conditions.

iii) Applying the extending operation to the equation $(x - t_1)\frac{du}{dx} = \lambda u$ with the Riemann scheme $\begin{Bmatrix} x = \infty & x = t_1 \\ -\lambda & \lambda \end{Bmatrix}$, we have a Gauss hypergeometric system with

the Riemann scheme $\begin{Bmatrix} x=\infty & x=t_1 & x=t_2 \\ -\rho_1 & 0 & 0 \\ -\rho_2 & \lambda & \rho_1+\rho_2-\lambda \end{Bmatrix}$, which is linearly irreducible. Here λ, ρ_1 and ρ_2 are any complex numbers satisfying $\rho_1\rho_2\lambda(\rho_1-\lambda)(\rho_2-\lambda)\neq 0$.

Theorem 3.8 follows from Theorem 2.2 and the argument just before Theorem 3.8. We will examine the Riemann scheme of (3.15). In fact Theorem 2.2 proves that the operation $M^{p+1}_{(0,\dots,0,\rho_2-\rho_1)}\circ T_{(p+1,\infty)}\circ mc_{-\rho_1}$ transforms (3.17) to

$$\begin{Bmatrix} x=\infty & x=t_1 & \cdots & x=t_p & x=t_{p+1} \\ [\rho_1-\rho_2]_{(n-m_1)} & [0]_{(n-n_1-m_1)} & \cdots & [0]_{(n-n_p-m_1)} & [\rho_2-\mu_2]_{(m_2)} \\ & [\lambda_{1,1}-\rho_1]_{(\ell_{1,1})} & \cdots & [\lambda_{p,1}-\rho_1]_{(\ell_{p,1})} & [\rho_2-\mu_3]_{(m_3)} \\ & \vdots & & \vdots & \vdots \\ & [\lambda_{1,r_1}-\rho_1]_{(\ell_{1,r_1})} & \cdots & [\lambda_{p,r_q}-\rho_1]_{(\ell_{p,r_q})} & [\rho_2-\mu_q]_{(m_q)} \end{Bmatrix}$$

and then the farther operation mc_{ρ_1} to this gives (3.19) because $\rho_2-\mu_2=0$ and $\rho_1-\rho_2\neq\rho_1$.

4. Yokoyama's restricting operation

Yokoyama's restriction is the inverse of his extension and we have the following theorem.

Theorem 4.1 (Restricting operation)**.** *Let* (1.6) *be a linearly irreducible Fuchsian system of ONF. Under the notation in Theorem 3.8 we assume $q=2$ and*

$$\mu_1+\mu_2\neq\lambda_{p,\nu}\qquad(\nu=1,\dots,r_p). \tag{4.1}$$

Then $mc_{\mu_1}\circ T_{(p,\infty)}\circ M^p_{(0,\dots,0,\mu_1-\mu_2)}\circ mc_{-\mu_1}(A_1,\dots,A_p)$ defines a linearly irreducible Fuchsian system

$$\big(xI_{\check n}-\check T\big)\frac{du}{dx}=\check A u \tag{4.2}$$

of ONF, whose Riemann scheme is

$$\begin{Bmatrix} x=\infty & x=t_1 & \cdots & x=t_{p-1} \\ [-\mu_1]_{(m_1-n_p)} & [0]_{(\check n-n_1)} & \cdots & [0]_{(\check n-n_{p-1})} \\ [-\mu_2]_{(m_2-n_p)} & [\lambda_{1,1}]_{(\ell_{1,1})} & \cdots & [\lambda_{p-1,1}]_{(\ell_{p-1,1})} \\ [\lambda_{p,1}-\mu_1-\mu_2]_{(\ell_{p,1})} & \vdots & & \vdots \\ \vdots & & & \\ [\lambda_{p,r_p}-\mu_1-\mu_2]_{(\ell_{p,r_p})} & [\lambda_{1,r_1}]_{(\ell_{1,r_1})} & \cdots & [\lambda_{p-1,r_{p-1}}]_{(\ell_{p-1,r_{p-1}})} \end{Bmatrix}. \tag{4.3}$$

Here the rank of the resulting system equals $\check n=n-n_p=n_1+\cdots+n_{p-1}$ and

$$\check T=\begin{pmatrix} t_1I_{n_1} & & \\ & \ddots & \\ & & t_{p-1}I_{n_{p-1}} \end{pmatrix},\quad \check A=\begin{pmatrix} A_{11} & \cdots & A_{1,p-1} \\ \vdots & \vdots & \vdots \\ A_{p-1,1} & \cdots & A_{p-1,p-1} \end{pmatrix}. \tag{4.4}$$

Proof. Suppose $q=2$. The operation $M^p_{(0,\dots,0,\mu_1-\mu_2)}\circ mc_{-\mu_1}$ transforms (3.17) to

$$\begin{Bmatrix} x=\infty & x=t_1 & \cdots & x=t_{p-1} & x=t_p \\ [0]_{(m_2)} & [0]_{(n-n_1-m_1)} & \cdots & [0]_{(n-n_{p-1}-m_1)} & [\mu_1-\mu_2]_{(n-n_p-m_1)} \\ & [\lambda_{1,1}-\mu_1]_{(\ell_{1,1})} & \cdots & [\lambda_{p-1,1}-\mu_1]_{(\ell_{p-1,1})} & [\lambda_{p,1}-\mu_2]_{(\ell_{p,1})} \\ & \vdots & & \vdots & \vdots \\ & [\lambda_{1,r_1}-\mu_1]_{(\ell_{1,r_1})} & \cdots & [\lambda_{p-1,r_q}-\mu_1]_{(\ell_{p-1,r_{p-1}})} & [\lambda_{p,r_q}-\mu_2]_{(\ell_{p,r_p})} \end{Bmatrix}$$

and the farther application $mc_{\mu_1}\circ T_{(p,\infty)}$ to the above gives (4.3) because $\mu_1\neq\mu_1-\mu_2$ and $\mu_1\neq\lambda_{p,\nu}-\mu_2$ for $\nu=1,\dots,r_p$, which corresponds to a system of ONF

as is claimed in Lemma 2.4. Here we note that the rank of the resulting system equals

$$\begin{aligned} m_2 &- \big((n-n_1-m_1)+\cdots+(n-n_{p-1}-m_1)+0-(p-2)m_2\big)\\ &= n_1+\cdots+n_{p-1}-(p-1)n+(p-1)(m_1+m_2)\\ &= n-n_p. \end{aligned}$$

Since the restricting operation defined in the theorem is the inverse of the extending operation in Theorem 3.8, we have (4.4). □

Remark 4.2. Suppose (4.1) is not valid. If we apply E_τ with generic $\tau \in \mathbb{C}$ to the original system of ONF preceding to the restriction, the resulting restriction satisfies (4.1). Note that mc_τ corresponding to the transformations of A, $\lambda_{j,\nu}$ and μ_k to $A+\tau$, $\lambda_{j,\nu}+\tau$ and $\mu_k+\tau$, respectively (cf. Corollary 3.4).

Remark 4.3. i) The extension and restriction give transformations between linearly irreducible systems of ONF. These operations do not change their indices of rigidity.

ii) The system (1.6) is called *strongly reducible* by [Yo] if there exists a non-trivial proper subspace of $\mathbb{C}^n$ which is invariant under T and A. It is shown there that if the system is not strongly reducible, this property is kept by these operations.

5. Equivalence of algorithms

In this section the system (1.1) of SCF defined by $\mathbf{A}=(A_1,\dots,A_p)\in M(n,\mathbb{C})^p$ is identified with the system defined by $\mathbf{B}\in M(n,\mathbb{C})^p$ if $\mathbf{A}\sim\mathbf{B}$ and then the system is ONF if a representative of $\mathbf{A}$ has the form (1.9).

Proposition 5.1. *Let* $\mathbf{A}=(A_1,\dots,A_p)\in M(n,\mathbb{C})^p$ *with* (2.8) *and* (2.9). *Then* $mc_\lambda(\mathbf{A})$ *is of ONF if and only if* λ *is not the eigenvalue of* $A_0 := -A_1-\cdots-A_p$. *In this case the corresponding system of ONF is given by* (2.25).

Proof. Putting $d=\dim\ker A_1+\cdots+\dim\ker A_p+\dim\ker(A_0-\lambda)-(p-1)n$, the rank of the system defined by $mc_\lambda(\mathbf{A})$ equals $n-d$. Lemma 1.1 implies that $mc_\lambda(\mathbf{A})$ is of ONF if and only if $\sum_{j=1}^p\big(n-\dim\ker A_j\big)=n-d$, which means $\dim\ker(A_0-\lambda)=0$. □

Definition 5.2. We denote by $E^p_{\rho_1,\rho_2}$ the extending operation of the system of ONF given in Theorem 3.8 and by R^p the restricting operation given in Theorem 4.1. Then the restricting operation R^p_j is defined by $R^p\circ T_{(j,p)}$ for $j=1,\dots,p$. Here (j,p) is the transposition of indices j and p (cf. Definition 3.6). Note that the restricting operation is defined only when $q=2$.

We have proved that the extension and the restriction of the system of ONF is realized by suitable combinations of additions, middle convolutions and the automorphism of $\mathbb{P}^1(\mathbb{C})$ written by $T_{(p+1,\infty)}$ and T_σ (cf. Definition 3.6).

In fact, we have the following equalities for operations to linearly irreducible systems (1.6) of ONF.

$$E^p_{\rho_1,\rho_2}=mc_{\rho_1}\circ M^{p+1}_{(0,\dots,0,\rho_2-\rho_1)}\circ T_{(p+1,\infty)}\circ mc_{-\rho_1}, \tag{5.1}$$

$$R^p=mc_{\mu_1}\circ T_{(p,\infty)}\circ M^p_{(0,\dots,0,\mu_1-\mu_2)}\circ mc_{-\mu_1}, \tag{5.2}$$

$$R^{p+1}\circ E^p_{\rho_1,\rho_2}=\mathrm{id}\,. \tag{5.3}$$

Here μ_1 and $\mu_2\in\mathbb{C}$ are determined by

$$(A-\mu_1)(A-\mu_2)=0. \tag{5.4}$$

Lemma 5.3. *We have the following relations for $j = 1, \ldots, p$.*

$$R_j^{p+1} \circ E_\epsilon \circ E^p_{\rho_1,\rho_2} = mc_{\rho_1+\epsilon} \circ M^p_{(0,\ldots,0,\underset{\frown\atop j}{\rho_2-\rho_1},0,\ldots,0)} \circ T_{(j,\infty)} \circ mc_{-\rho_1}, \tag{5.5}$$

$$\operatorname{ord} R_j^{p+1} \circ E_\epsilon \circ E^p_{\rho_1,\rho_2}(\mathbf{A}) = \operatorname{ord}\mathbf{A} + \dim \operatorname{Im}(A-\rho_1)(A-\rho_2) - \dim \operatorname{Im} A_j, \tag{5.6}$$

$$\begin{aligned} &R_j^{p+1} \circ E_{\rho_1+\epsilon,\rho_1+\rho_2+\rho_3+\epsilon} \circ R_j^{p+1} \circ E_\epsilon \circ E^p_{\rho_1,\rho_2} \\ &\quad = mc_{\rho_1+\epsilon} \circ M^p_{(0,\ldots,0,\underset{\frown\atop j}{\rho_1+\rho_3},0,\ldots,0)} \circ mc_{-\rho_1}. \end{aligned} \tag{5.7}$$

Here ρ_1 and ρ_2 are any non-zero complex numbers and ϵ is a generic complex number (cf. Remark 4.2) and $\operatorname{ord}\mathbf{A}$ *denotes the rank of the corresponding system* (1.1) *of SCF.*

Proof. We may assume $j = 1$. It follows from (5.1) and (5.2) that

$$\begin{aligned} &R_1^{p+1} \circ E_\epsilon \circ E^p_{\rho_1,\rho_2} \\ &= mc_{\rho_1+\epsilon} \circ T_{(p+1,\infty)} \circ M^{p+1}_{(0,\ldots,0,\rho_1-\rho_2)} \circ mc_{-\rho_1-\epsilon} \circ T_{(1,p+1)} \circ mc_\epsilon \\ &\quad \circ mc_{\rho_1} \circ M^{p+1}_{(0,\ldots,0,\rho_2-\rho_1)} \circ T_{(p+1,\infty)} \circ mc_{-\rho_1} \\ &= mc_{\rho_1+\epsilon} \circ T_{(p+1,\infty)} \circ M^{p+1}_{(0,\ldots,0,\rho_1-\rho_2)} \circ T_{(1,p+1)} \circ M^{p+1}_{(0,\ldots,0,\rho_2-\rho_1)} \circ T_{(p+1,\infty)} \\ &\quad \circ mc_{-\rho_1} \\ &= mc_{\rho_1+\epsilon} \circ T_{(p+1,\infty)} \circ M^{p+1}_{(\rho_2-\rho_1,0,\ldots,0,\rho_1-\rho_2)} \circ T_{(1,p+1)} \circ T_{(p+1,\infty)} \circ mc_{-\rho_1} \\ &= mc_{\rho_1+\epsilon} \circ M^p_{(\rho_2-\rho_1,0,\ldots,0)} \circ T_{(1,\infty)} \circ mc_{-\rho_1} \end{aligned}$$

and therefore

$$\begin{aligned} &R_1^{p+1} \circ E^p_{\rho_1+\epsilon,\rho_1+\rho_2+\rho_3+\epsilon} \circ R_1^{p+1} \circ E_\epsilon \circ E^p_{\rho_1,\rho_2} \\ &= mc_{\rho_1+\epsilon} \circ M^p_{(\rho_2+\rho_3,0,\ldots,0)} \circ T_{(1,\infty)} \circ mc_{-\rho_1-\epsilon} \\ &\quad \circ mc_{\rho_1+\epsilon} \circ M^p_{(-\rho_1+\rho_2,0,\ldots,0)} \circ T_{(1,\infty)} \circ mc_{-\rho_1} \\ &= mc_{\rho_1+\epsilon} \circ M^p_{(\rho_1+\rho_3,0,\ldots,0)} \circ mc_{-\rho_1}. \end{aligned}$$

The equality (5.6) follows from Theorem 3.8 and Theorem 4.1. □

We show Riemann schemes related to (5.7).

Remark 5.4. By the extension $E^p_{-\lambda_{0,1},-\lambda_{0,2}}$ we have

$$\begin{Bmatrix} x=\infty & x=t_1 & \cdots & x=t_p \\ [\lambda_{0,1}]_{(m_{0,1})} & [0]_{(m_{1,1})} & \cdots & [0]_{(m_{p,1})} \\ [\lambda_{0,2}]_{(m_{0,2})} & [\lambda_{1,2}]_{(m_{1,2})} & \cdots & [\lambda_{p,2}]_{(m_{p,2})} \\ \vdots & \vdots & & \vdots \end{Bmatrix} \mapsto$$

$$\begin{Bmatrix} x=\infty & x=t_1 & \cdots & x=t_{p+1} \\ [\lambda_{0,1}]_{(n-m_{0,2})} & [0]_{(m_{1,1}+n-m_{0,1}-m_{0,2})} & \cdots & [0]_{(n)} \\ [\lambda_{0,2}]_{(n-m_{0,1})} & [\lambda_{1,2}]_{(m_{1,2})} & \cdots & [\lambda_{0,3}-\lambda_{0,1}-\lambda_{0,2}]_{(m_{0,3})} \\ & \vdots & & \vdots \end{Bmatrix}.$$

Here $n = m_{j,1} + m_{j,2} + \cdots$ and $m_{1,1} + \cdots + m_{p,1} = (p-1)n$. By applying the restriction $R_1^{p+1} \circ E_\epsilon$ to this result we have

$$\left\{\begin{matrix} x=\infty & x=t_1 & x=t_2 & \cdots \\ [\lambda_{0,1}-\epsilon]_{(m_{1,1}-m_{0,2})} & [0]_{(m_{1,1})} & [0]_{(m_{2,1}-m_{0,1}-m_{0,2}+m_{1,1})} & \cdots \\ [\lambda_{0,2}-\epsilon]_{(m_{1,1}-m_{0,1})} & [\lambda_{0,3}-\lambda_{0,1}-\lambda_{0,2}+\epsilon]_{(m_{0,3})} & [\lambda_{2,2}+\epsilon]_{m_{2,2}} & \cdots \\ [\lambda_{1,2}+\lambda_{0,1}+\lambda_{0,2}-\epsilon]_{(m_{1,2})} & \vdots & \vdots & \cdots \\ [\lambda_{1,3}+\lambda_{0,1}+\lambda_{0,2}-\epsilon]_{(m_{1,3})} & & & \\ \vdots & & & \end{matrix}\right\}$$

whose rank equals $n - (m_{0,1} + m_{0,2} - m_{1,1})$. By applying the extending operation $E^p_{-\lambda_{0,1}+\epsilon, -\lambda_{1,2}-\lambda_{0,1}-\lambda_{0,2}+\epsilon}$ to what we obtained we have

$$\left\{\begin{matrix} x=\infty & x=t_1 \\ [\lambda_{0,1}-\epsilon]_{(n-m_{0,1}-m_{0,2}+m_{1,1}-m_{1,2})} & [0]_{(n-m_{0,1}+m_{1,1}-m_{1,2})} \\ [\lambda_{1,2}+\lambda_{0,1}+\lambda_{0,2}-\epsilon]_{(n-m_{0,1})} & [\lambda_{0,3}-\lambda_{0,1}-\lambda_{0,2}+\epsilon]_{(m_{0,3})} \\ & [\lambda_{0,4}-\lambda_{0,1}-\lambda_{0,2}+\epsilon]_{(m_{0,4})} \\ & \vdots \end{matrix}\right.$$

$$\left.\begin{matrix} x=t_2 & \cdots & x=t_{p+1} \\ [0]_{(n-2m_{0,1}-m_{0,2}+m_{1,1}-m_{1,2}+m_{2,1})} & \cdots & [0]_{(n-m_{0,1}-m_{0,2}+m_{1,1})} \\ [\lambda_{2,2}+\epsilon]_{(m_{2,2})} & \cdots & [-2\lambda_{0,1}-\lambda_{1,2}+\epsilon]_{(m_{1,1}-m_{0,1})} \\ \vdots & \cdots & [\lambda_{1,3}-\lambda_{1,2}-\lambda_{0,1}+\epsilon]_{(m_{1,3})} \\ & & \vdots \end{matrix}\right\}$$

and by applying the restriction R_1^{p+1} to this result we finally have

$$\left\{\begin{matrix} x=\infty & x=t_1 & x=t_2 & \cdots \\ [\lambda_{0,1}-\epsilon]_{(m_{0,1}-d)} & [0]_{(m_{1,1})} & [0]_{(m_{2,1}-d)} & \cdots \\ [\lambda_{0,2}+\lambda_{0,1}+\lambda_{1,2}-\epsilon]_{(m_{0,2})} & [-2\lambda_{0,1}-\lambda_{1,2}+\epsilon]_{(m_{1,1}-m_{0,1})} & [\lambda_{2,2}+\epsilon]_{(m_{2,2})} & \cdots \\ [\lambda_{0,3}+\lambda_{0,1}+\lambda_{1,2}-\epsilon]_{(m_{0,3})} & [\lambda_{1,3}-\lambda_{1,2}-\lambda_{0,1}+\epsilon]_{(m_{1,3})} & \vdots & \cdots \\ [\lambda_{0,4}+\lambda_{0,1}+\lambda_{1,2}-\epsilon]_{(m_{0,4})} & [\lambda_{1,4}-\lambda_{1,2}-\lambda_{0,1}+\epsilon]_{(m_{1,4})} & & \\ \vdots & \vdots & & \end{matrix}\right\}$$

with $d = m_{0,1} - m_{1,1} + m_{1,2}$.

Theorem 5.5. *Suppose* $\mathbf{A} = (A_1, \ldots, A_p) \in M(n, \mathbb{C})^p$ *is irreducible and suppose* $\mathbf{B} = (B_1, \ldots, B_p) \in M(n, \mathbb{C})^p$ *is obtained from* $\mathbf{A}$ *by a finite iteration of additions, middle convolutions and operations* $T_{(p,\infty)}$ *and* T_σ *in Definition 3.6.*

Let α *and* β *be generic complex numbers so that* $mc_\alpha(\mathbf{A})$ *and* $mc_\beta(\mathbf{B})$ *are of ONF. Then* $mc_\beta(\mathbf{B})$ *can be obtained from* $mc_\alpha(\mathbf{A})$ *by a finite iteration of the suitable operations* $R_j^{p+1} \circ E_\epsilon \circ E^p_{\rho_1,\rho_2}$*, namely, extensions, restrictions and generic Euler transformations. Here* $\alpha = 0$ *is generic if* $\mathbf{A}$ *is of ONF.*

Proof. The theorem follows from Lemma 5.3 since $R_j^{p+1} \circ E^p_{\rho_1,\rho_1} = T_{(j,\infty)}$, $T_{(i,j)} = T_{(j,\infty)} \circ T_{(i,\infty)} \circ T_{(j,\infty)}$, $M_\mu \circ M_{\mu'} = M_{\mu+\mu'}$, $mc_\lambda \circ mc_{\lambda'} = mc_{\lambda+\lambda'}$ and $mc_0 = id$. □

6. Reduction process

For the system (1.1) of SCF the *spectral type* of $\mathbf{A} = (A_1, \ldots, A_p)$ denoted by $\operatorname{spt}\mathbf{A}$ is the tuple of $p+1$ partitions of n

$$\operatorname{spt}\mathbf{A} := \mathbf{m} = (m_{0,1}, \ldots, m_{0,n_0}; m_{1,1}, \ldots, m_{1,n_1}; \ldots; m_{p,1}, \ldots, m_{p,n_p}) \tag{6.1}$$

under the notation (2.6). This tuple may be expressed by

$$m_{0,1} \cdots m_{0,n_0}, m_{1,1} \cdots m_{1,n_1}, \cdots, m_{p,1} \cdots m_{p,n_p} \tag{6.2}$$

and in this case (2.3) shows

$$\operatorname{idx}\mathbf{A} = \sum_{\substack{1\le\nu\le n_j\\0\le j\le p}} m_{j,\nu}^2 - (p-1)(\operatorname{ord}\mathbf{A})^2. \tag{6.3}$$

We put $n_j = 1$ and $m_{j,1} = \operatorname{ord}\mathbf{m} := m_{0,1} + \cdots + m_{0,n_0}$ if $j > p$. Moreover we put $m_{j,\nu} = 0$ if $j > n_j$.

For $p+1$ non-negative integers $\tau = (\tau_0, \ldots, \tau_p)$ we define

$$d_\tau(\mathbf{m}) := m_{0,\tau_0} + \cdots + m_{p,\tau p} - (p-1)\operatorname{ord}\mathbf{A} \tag{6.4}$$

and $\tau(\mathbf{m}) = \big(\tau(\mathbf{m})_0, \ldots, \tau(\mathbf{m})_p\big)$ so that

$$m_{j,\tau(\mathbf{m})_j} \ge m_{j,\nu} \qquad (\nu = 1, \ldots, n_j,\ j = 0, \ldots, p). \tag{6.5}$$

Moreover we put

$$d_{\max}(\mathbf{m}) := d_{\tau(\mathbf{m})}(\mathbf{m}). \tag{6.6}$$

Suppose $\mathbf{A}$ is irreducible. Put

$$mc_{\max}(\mathbf{A}) := mc_{\lambda_{0,\tau(\mathbf{m})_0}+\cdots+\lambda_{p,\tau(\mathbf{m})_p}} \circ M_{(-\lambda_{1,\tau(\mathbf{m})_1},\ldots,-\lambda_{p,\tau(\mathbf{m})_p})}(\mathbf{A}) \tag{6.7}$$

under the notation (2.6). If $n > 1$, then Theorem 2.2 proves

$$\begin{cases}\operatorname{spt} mc_{max}(\mathbf{A}) = \partial_{max}(\mathbf{m}) := (\ldots; m'_{j,1}, \ldots, m'_{j,n_j}; \ldots)\\ m'_{j,\nu} = m_{j,\nu} - d_{\max}(\mathbf{m})\delta_{\nu,\tau(\mathbf{m})_j} \qquad (\nu = 1, \ldots, n_j,\ j = 0, \ldots, p)\end{cases}, \tag{6.8}$$

$$\operatorname{ord}\partial_{max}(\mathbf{m}) = \operatorname{ord}\mathbf{m} - d_{max}(\mathbf{m}). \tag{6.9}$$

If $\mathbf{A}$ is rigid, namely, $\operatorname{idx}\mathbf{m} = 2$, then we have $d_{max}(\mathbf{m}) > 0$ because

$$\operatorname{idx}\mathbf{m} + \sum_{j=0}^{p}\sum_{\nu=1}^{n_j}(m_{j,\tau_j} - m_{j,\nu})\cdot m_{j,\nu} = \left(\sum_{j=0}^{p} m_{j,\tau_j} - (p-1)\operatorname{ord}\mathbf{m}\right)\cdot\operatorname{ord}\mathbf{m}$$

and thus we have $\operatorname{ord} mc_{max}(\mathbf{A}) < \operatorname{ord}\mathbf{A}$. Hence if the system of SCF is linearly irreducible and rigid, the system is connected to a rank 1 system by a finite iteration of additions and middle convolutions and conversely any linearly irreducible system of SCF is constructed from a rank 1 system by a finite iteration of additions and middle convolutions (cf. [Kz], [Ko], [DR], [O2]).

Since any rank 1 system is transformed into ONF by a suitable addition, Theorem 5.5 implies the following theorem, which is given in [Yo, Theorem 4.6] when the parameters $\lambda_{i,\nu}$ and μ_j are generic.

Theorem 6.1. *Any linearly irreducible rigid system of ONF is connected to a rank 1 system of ONF by a finite iteration of extensions, restrictions and generic Euler transformations.*

Remark 6.2. i) For a given $\mathbf{A} \in M(n,\mathbb{C})^p$, if there exists j with $d_{max}(\operatorname{spt}\mathbf{A}) > m_{j,\tau(\operatorname{spt}\mathbf{A})_j}$, $\mathbf{A}$ is not irreducible. This is a consequence of Theorem 3.8.

ii) It follows from Proposition 5.1 that $mc_{\max}(\mathbf{A})$ is not of ONF for any linearly irreducible system (1.1) of SCF.

iii) In virtue of Lemma 6.3 a more explicit construction of the reduction process within ONF using extensions, restrictions and generic Euler transformations is obtained as follows.

Put $\mathbf{m} = \operatorname{spt}(\mathbf{A})$ for a linearly irreducible system (1.6) of ONF. Assume that $\mathbf{m}$ satisfies the assumption of Lemma 6.3 and $\lambda_{j,1} = 0$ for $j = 1, \ldots, p$. Then Lemma 6.3 assures that we can find $j \ge 1$ with

$$m_{0,1} - m_{j,1} + m_{j,2} > 0 \tag{6.10}$$

because $d_{max}(\mathbf{m}) = m_{0,1}$. Applying the operation (5.7) with $\rho_1 = \lambda_{0,1}$, $\rho_2 = \lambda_{0,2}$ and $\rho_3 = \lambda_{1,2}$, it follows from Remark 5.4 that the resulting $\mathbf{A}'$ satisfies

$$\operatorname{ord}\mathbf{A}' = \operatorname{ord}\mathbf{A} - m_{0,1} + m_{j,1} - m_{j,2} < \operatorname{ord}\mathbf{A}. \tag{6.11}$$

iv) The existence of $j \geq 1$ satisfying (6.10) is given by [Yo, Lemma 4.2] when the rigidity index of the system of ONF equals 2. Note that any linearly irreducible rigid system of SCF with rank > 1 always satisfies the assumption of Lemma 6.3.

Lemma 6.3. *Let* $\mathbf{m}$ *be a spectral type of a linearly irreducible system* (1.1) *of SCF with* $\operatorname{ord}\mathbf{m} > 1$. *Put* $\mathbf{m}' = \partial_{max}(\mathbf{m})$. *We may assume* $m_{j,1} \geq m_{j,2} \geq \cdots \geq m_{j,n_j}$. *If* $d_{max}(\mathbf{m}) > 0$ *and* $d_{max}(\mathbf{m}') > 0$, *then*

$$\sum_{j=0}^{p} \max\{0, d_{max}(\mathbf{m}) - (m_{j,1} - m_{j,2})\} > d_{max}(\mathbf{m}). \tag{6.12}$$

Proof. Put $d = d_{max}(\mathbf{m})$. Since $\max\{m'_{j,1}, \ldots, m'_{j,n_j}\} = \max\{m_{j,2}, m_{j,1} - d\}$, the assumption implies

$$\sum_{j=0}^{p} \max\{m_{j,2}, m_{j,1} - d\} > (p-1)\operatorname{ord}\mathbf{m}' = (p-1)(n-d).$$

Hence we have

$$\begin{aligned}\sum_{j=0}^{p} \max\{d - (m_{j,1} - m_{j,2}), 0\} &> (p-1)(n-d) - \sum_{j=1}^{p}(m_{j,1} - d) \\ &= (p-1)(n-d) - (p-1)n + pd = d.\end{aligned}$$

□

A linearly irreducible system (1.1) of SCF satisfying $d_{max}(\operatorname{spt}\mathbf{A}) \leq 0$ is called *basic*, which is not rigid and not of ONF. It is known that the basic systems of SCF with different spectral types cannot be connected by any iteration of middle convolutions, additions, $T_{j,\infty}$ and T_σ. Moreover there exist a finite number of basic systems with a fixed index of rigidity and an indivisible spectral type (cf. [CB], [O2, Proposition 8.1]). Here $\mathbf{m} = (\ldots; m_{j,1}, \ldots, m_{j,n_j}; \ldots)$ is indivisible if there doesn't exist a non-trivial common divisor of $\{m_{j,\nu}\,;\, j = 0, 1, \ldots,\ \nu = 1, 2, \ldots\}$ and two tuples are identified if a permutation of indices j and permutations of indices ν within the same j transform one of the two into the other.

It is shown by [CB] that the basic systems with a given index of rigidity correspond to the positive imaginary roots with a fixed norm in the closure of a negative Weyl chamber of a Kac-Moody root system with a star-shaped Dynkin diagram (cf. [Kc], [O2, §7]). Any linearly irreducible system of SCF which is not rigid is connected to a basic system by an iteration of mc_{max} and therefore we have the following theorem.

Theorem 6.4. *By a finite iteration of extensions, restrictions and generic Euler transformations, any linearly irreducible system of ONF which is not rigid is connected to a system of ONF transformed by a middle convolution of a basic system of SCF (cf. Proposition 5.1).*

We will give some examples.

Example 6.5. There exist 4 different spectral types of basic systems with index of rigidity 0 (cf. [Ko2], [O2, Proposition 8.1]):

type	ord	basic system	ord	ONF
$\tilde{D}_4$	2	11,11,11,11	3	111,21,21,21
$\tilde{E}_6$	3	111,111,111	4	1111,211,211
$\tilde{E}_7$	4	1111,1111,22	5	11111,2111,32
$\tilde{E}_8$	6	111111,222,33	7	1111111,322,43

The following is the list of spectral types of basic systems with index of rigidity -2 (cf. [O2, Proposition 8.4]):

ord	basic system	ord	ONF	
2	11,11,11,11,11	4	211,31,31,31,31	
3	111,111,21,21	4	1111,211,31,31	
4	1111,22,22,31	5	11111,32,32,41	
4	1111,1111,211	5	11111,2111,311	
4	211,22,22,22	6	2211,42,42,42	222,411,42,42
5	11111,221,221	6	111111,321,321	
5	11111,11111,32	6	111111,21111,42	
6	111111,2211,33	7	1111111,3211,43	
6	2211,222,222	8	22211,422,422	2222,422,4211
8	11111111,332,44	9	111111111,432,54	
8	22211,2222,44	10	222211,4222,64	22222,42211,64
10	22222,3331,55	12	222222,5331,75	
12	2222211,444,66	14	22222211,644,86	

Here we give the spectral types of systems of ONF with minimal rank corresponding to a basic system, which are not necessarily unique but transformed to each other by suitable iterations of extensions, restrictions and generic Euler transformations.

Definition 6.6. For a $(p+1)$-tuple $\mathbf{m} = (m_{j,\nu})$ of partitions of n we put

$$\operatorname{Oidx}\mathbf{m} := (p-1)\cdot \operatorname{ord}\mathbf{m} - \max_{0\le k\le p} \sum_{\substack{0\le j\le p\\ j\ne k}} \max\{m_{j,1}, m_{j,2}, \ldots\}. \tag{6.13}$$

We define that $\mathbf{m}$ is of *Okubo type* if $\operatorname{Oidx}\mathbf{m} = 0$.

Remark 6.7. Let $\mathbf{m}$ be the spectral type of a linearly irreducible system (1.1) of SCF. Then $\operatorname{Oidx}\mathbf{m} \ge 0$. Moreover it follows form Lemma 1.10 and Remark 1.2 that the system is equivalent to a system of ONF after applying a suitable addition (and $T_{(p,\infty)}$) if and only if $\operatorname{Oidx}\mathbf{m} = 0$.

If $\mathbf{m}$ is basic, then $\operatorname{Oidx}\mathbf{m} > 0$ and there exists a system of ONF with the minimal rank $\operatorname{ord}\mathbf{m} + \operatorname{Oidx}\mathbf{m}$ among the systems obtained from the original system (1.1) by a finite iteration of additions and middle convolutions (cf. Proposition 5.1).

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